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TITLE: MULTIGRID SEMI-IMPLICIT HYDRODYNAMICS REVISITED

AUTHOR(S): Joel E. Dendy

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Los Alamos
Los Alamos National Laboratory
Los Alamos, New Mexico 87545

MULTIGRID SEMI-IMPLICIT HYDRODYNAMICS REVISITED

Joel E. Dendy, Jr.

1. INTRODUCTION.

The multigrid method has for several years been very successful for simple equations like Laplace's equation on a rectangle. For more complicated situations, however, success has been more elusive. Indeed, there are only a few applications in which the multigrid method is now being successfully used in complicated production codes. The one with which we are most familiar is the application by Alcouffe [1] to TTDAMG, stemming from [2]. We are more familiar with this second application in which, for a set of test problems, TTDAMG ran seven to twenty times less expensively (on a CRAY-1 computer) than its best competitor. This impressive performance, in a field where a factor of two improvement is considered significant, encourages one to attempt the application of the multigrid method in other complicated situations.

The application discussed in this paper was actually attempted several years ago in [4]. In that paper the multigrid method was applied to the pressure iteration in three Eulerian and Lagrangian codes. The application to the Eulerian codes, both incompressible and compressible, was successful, but the application to the Lagrangian code was less so. The reason given for this lack of success in [4] was that the differencing for the pressure

equation in the Lagrangian code, SALE, was bad. For example, on a uniform grid with mesh size h, the approximation to the Laplacian Δ in SALE is the skewed Laplacian:

$$\Delta_h^{sk} p_{i,j} = \frac{1}{2h^2}(p_{i-1,j-1} + p_{i-1,j+1} + p_{i+1,j-1}$$

$$+ p_{i+1,j+1} - 4p_{i,j}) \; ; \qquad (1.1)$$

why this is a bad differencing is discussed below. In [4] the differencing for the pressure equation was changed so that on a uniform grid, the Laplacian was approximated by the discrete five point Laplacian:

$$\Delta_h p_{i,j} = \frac{1}{h^2}(p_{j-1,j} + p_{i+1,j} + p_{i,j-1}$$

$$+ p_{i,j+1} - 4p_{i,j}) \; ; \qquad (1.2)$$

when this change was made, we were able to apply the multigrid method successfully. In this paper, we examine again the application of multigrid to the pressure equation in SALE with the goal of succeeding this time without cheating.

2. MULTIGRID REVIEW.

To explain the difficulty with (1.1), it is helpful to review the multigrid method. Suppose that the equation LU = F is approximated on a grid G^M by

$$L^M U^M = F^M \quad . \qquad (2.1)$$

In the simplest form of the multigrid method, one constructs a sequence of grids $G^1, \ldots, G^M$ with corresponding mesh sizes $h_1, \ldots, h_M$, where $h_{i-1} = 2h_i$. One does a fixed number, IM, of relaxation sweeps (Gauss-Seidel, for example) on (2.1) and then drops down to grid G^{M-1} and the equation

$$L^{M-1}U^{M-1} = f^{M-1} = I_M^{M-1}(F^M - L^M v^M) \; , \qquad (2.2)$$

where v^{M-1} is to be the coarse grid approximation to $v^M \equiv U^M - u^M$, where $v^M = u^M$ is the last iterate on grid G^M, and where I_M^{M-1} is an interpolation operator from G^M to G^{M-1}. To solve equation (2.2) approximately, one resorts to recursion, taking ID relaxation sweeps on grid G^k before dropping down to grid G^{k-1}, $M-1 \geq k \geq 2$ and the equation

$$L^{k-1}v^{k-1} = f^{k-1} \equiv I_k^{k-1}(f^k - L^kv^k) \quad . \tag{2.3}$$

When grid G^1 is reached, the equation $L^1v^1 = f^1$ can either be solved directly or to some precision by iteration and $v^2 \leftarrow v^2 + I_1^2 v^1$ can be performed. Then one does IU relaxation sweeps on grid G^{k-1} before forming $v^k \leftarrow v^k + I_{k-1}^k v^{k-1}$, $3 \leq k \leq M$.

The motivation behind the multigrid method is that, on a given grid G^k, relaxation methods like Gauss-Seidel generally do a fine job of reducing high frequency components of the error but a poor job of reducing low frequency components. More specifically, let the $\theta = (\theta_1, \theta_2)$ Fourier component of the error functions v and $\bar{v}$ before and after a relaxation sweep on G^k be written as

$$\bar{v}_{\alpha,\beta} = \bar{A}_\theta e^{i(\theta_1\alpha+\theta_2\beta)} \quad \text{and} \quad v_{\alpha,\beta} = A_\theta e^{i(\theta_1\alpha+\theta_2\beta)}. \quad \text{Suppose}$$

that $\mu(\theta) = \left|\dfrac{\bar{A}_\theta}{A_\theta}\right|$ is appreciably less than one for components θ with $\dfrac{\pi}{2} \leq |\theta| = \max(|\theta_1|, |\theta_2|) \leq \pi$; then such components can be efficiently reduced by relaxation sweeps on G^k. Components θ with $0 \leq |\theta| \leq \dfrac{\pi}{2}$ are the ones which can be approximated on G^{k-1}; components θ with $\dfrac{\pi}{4} \leq |\theta| \leq \dfrac{\pi}{2}$ on G^k are mapped to components θ with $\dfrac{\pi}{2} \leq |\theta| \leq \pi$ on G^{k-1} and can be efficiently reduced by relaxation sweeps on G^{k-1}, while components θ with $0 \leq |\theta| \leq \pi$ on G^{k-1} are mapped to components θ with $0 \leq |\theta| \leq \dfrac{\pi}{2}$ on G^{k-2}, which are the ones which can be approximated on G^{k-2}. Recursion leads to G^1, which is assumed coarse enough either to solve directly or to iterate efficiently.

For the operator (1.2), $\bar{\mu} = \max\{\mu(\theta):\frac{\pi}{2}\leq|\theta|\leq\pi\} = .5$, and the multigrid method performs admirably [3]. However, for the operator (1.1), $\bar{\mu} = 0$ since $\mu(\pi,\pi) = 0$ for Δ_h^{sk}. Thus multigrid for Δ_h^{sk} performs no better for Δ_h^{sk} than simple relaxation. A numerical example of this bad performance is given in Section 4.

3.   <u>RAY MULTIGRID METHOD</u>.

The failure of multigrid for Δ_h^{sk} was a source of annoyance to Brandt, which motivated him to derive a cure, the description of which from [8] we now summarize. First, however, we need to be more specific about the choice of L^k, $k < M$, in (2.2) and (2.3). Assuming L^M to be positive definite, rewrite (2.1) as $(L^M)^{\frac{1}{2}}U^M = (L^M)^{-\frac{1}{2}}F^M$. Then given an approximation u^M to U^M, find V^{M-1} so as to minimize

$$E(V^{M-1}) = \left\| (L^M)^{\frac{1}{2}}(u^M + I_{M-1}^M V^{M-1}) - (L^M)^{-\frac{1}{2}}F^M \right\|_{G^M, L_2}. \quad \text{This}$$

minimization problem is equivalent to (2.2) if $L^{M-1} = I_M^{M-1}L^M I_{M-1}^M$ and $I_M^{M-1} = (I_{m-1}^M)^*$. A similar result holds by induction for L^k, $k < M-1$.

Let $S = \{\theta: \left|L^M(\theta)\right| << \left|L^M\right|, \quad \theta \leq \pi$, where $L^M(\theta)$ is the symbol of L^M (i.e., the function $L(\theta)$ which satisfies $L^M e^{i\theta \cdot x/h} = L^M(\theta)e^{i\theta \cdot x/h}$), $\left|L^M\right| = \max\limits_{|\theta|<\pi}\left|L^M(\theta)\right|$, and $h = h_M$. The Fourier modes $e^{i\theta \cdot x/h}$, $\theta\varepsilon S$ are the slowly convergent nodes in any reasonable local relaxation process. Hence, G^{M-1} should approximate these modes well. Attempt to write $R^M = f^M - L^M u^M$ and $V^M = U^M - u^M$ as

$$R^M(x) = \sum_{s\varepsilon S'} R_s^M(x)e^{is \cdot x/h}, \quad R_s^M \text{ smooth},$$

$$V^M(x) = \sum_{s\varepsilon S'} V_s^M(x)e^{is \cdot x/h}, \quad V_s^M \text{ smooth},$$

where $S' \subseteq S$ is a finite set such that if $\theta\varepsilon S$, then there exists $s\varepsilon S'$ such that $e^{i(\theta-s)\cdot x/h}$ is smooth, i.e., such

that $\left|\theta-s\right| \ll 1$. The role of G^{M-1} is to approximate V_s^M, $s \varepsilon S^1$, well, which should be possible since the V_s^M should be smooth after relaxation. For each $s \varepsilon S'$ we want an equation like

$$L_s^{M-1} V_s^{M-1} = I_{m,s}^{M-1} R^M \quad ; \tag{3.1}$$

to do this, consider

$$E(\{V_s^{M-1}\}) =$$

$$\left\| (L^M)^{\frac{1}{2}} (u^M + \sum_{s \varepsilon S'} e^{is \cdot x/h} I_{M-1,s}^M V_s^{M-1}) - (L^M)^{-\frac{1}{2}} F^M \right\|_{G^M, L_2}$$

$$= \left\| (L^M)^{\frac{1}{2}} (\sum_{s \varepsilon S'} e^{is \cdot x/h} I_{M-1,s}^M V_s^{M-1}) - (L^M)^{-\frac{1}{2}} R^M \right\|_{G^M, L_2}$$

$$\leq \sum_{s \varepsilon S'} \left\| (L^M)^{\frac{1}{2}} e^{is \cdot x/h} I_{M-1,s}^M V_s^{M-1} - (L^M)^{-\frac{1}{2}} e^{is \cdot x/h} R_s^M \right\|_{G^M, L_2} .$$

Minimization of each term in the last sum leads to (1.6) with $L_s^{M-1} = (I_{M-1,s}^M)^* L^M I_{M-1,s}^M$, $I_{M,s}^{M-1} = (I_{M-1},s)^*$, and $I_{M-1,s}^M = e^{is \cdot x/h} I_{M-1}^M$.

After solution of (3.1), the correction $u^M \leftarrow u^M + \sum_{s \varepsilon S'} I_{M-1,s}^M V_s^{M-1}$ is made. Of course, by recursion (3.1) can be solved approximately for each $s \varepsilon S'$ by relaxation and by construction of a set S_s' consisting of the slowly converging modes for relaxation on (3.1).

4. **APPLICATION OF THE RAY MULTIGRID METHOD TO Δ_h^{sk}.**

In this section we consider application of the ray method to the equation

$$-\Delta u + 10^{-4} u = F \text{ in } \Omega = (0, .96) \times (0, .96) \tag{4.1}$$

$$\frac{\partial u}{\partial v} = 0 \text{ on } \partial \Omega \quad .$$

The reason we consider (4.1) is that it is a model equation for the problem in SALE considered in [4], which had zero Neumann boundary conditions and a lower order term with a small multiple. The discrete approimation we consider is cell-centered as in SALE. At the i,j$\underline{\text{th}}$ cell

center, the discrete equation we consider is

$$-\Delta_h^{sk} U_{i,j}^M + 10^{-4} U_{i,j}^M = \cos(25\pi(i-2)h)\cos(25\pi(j-2)h) ,$$

$$2 \le i \le 25, \quad 2 \le j \le 25 . \qquad (4.2a)$$

Here $h = h_M = .04$, and the right hand side is chosen to be rich in the (π,π) frequency.

There are at least two possible approximations to the boundary conditions, which we will refer to as the "finite element" and "finite difference" approximation. The "finite element" approximation (so called because it results from using piecewise bilinear elements on quadrilaterals with midpoint quadrature) we illustrate by giving two typical cases: at $(2,2)$, $-\Delta_h^{sk} U_{i,j} = \frac{1}{2h^2}(-U_{i+1,j+1} + U_{i,j})$, and at $(2,j)$, $2 < j < 25$, $-\Delta_h^{sk} U_{i,j} = \frac{1}{2h^2}(-U_{i+1,j+1} - U_{i+1,j-1} + 2U_{i,j})$. The "finite difference" boundary approximation to the boundary condition can be derived by using fictitious cells, writing down a difference approximation to $\frac{\partial u}{\partial v} = 0$ (e.g., $\frac{U_{1,j} - U_{2,j}}{h} = 0$, $2 \le j \le 25$) and then eliminating the fictitious cells in terms of the interior cells. Two typical cases are:

at $(2,2)$,

$$-\Delta_h^{sk} U_{i,j} = \frac{1}{2h^2}(-U_{i,j+1} - U_{i+1,j} - U_{i+1,j+1} + 3U_{i,j}) \quad (4.2b)$$

and at $(2,j)$, $2 < j < 25$,

$$-\Delta_h^{sk} U_{i,j} = \frac{1}{2h^2}(-U_{i,j-1} - U_{i,j+1} - U_{i+1,j+1}$$

$$-U_{i+1,j-1} + 4U_{i,j}) . \qquad (4.2c)$$

For a uniform grid and $-\Delta$, the differencing in SALE reduces to the above "finite difference" approximation on the boundary; hence, it is of more interest to us. Also, the "finite element" boundary approximation annihilates the (π,π) frequency, but the "finite difference" boundary approximation does not. Not annihilating the (π,π)

frequency is a desirable feature since it is a partial cure for the "hourglass" instability that is troublesome in Lagrangian codes like SALE. (In [7], it is claimed that the "finite difference" boundary approach is a total cure for the "hourglass" instability. In fact, for some problems, it is still necessary to smooth the "hourglass" frequency a little; this is referred to as an "alternate node coupler" and is discussed further in Sec. 6.)

Before we can describe the numerical results for (4.1) we must describe the operators I_{k-1}^k. We assume G^{k-1} is every other grid point of G^k. (See Fig. 1.)
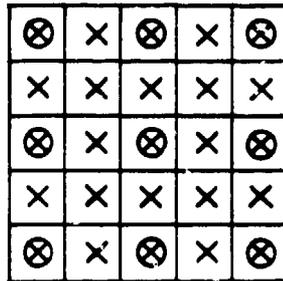

Fig. 1. Two cell-centered grids in which coarse grid unknowns are every other fine grid unknown.

Suppose that (IF,JF) ε G^k is the same point as (IC,JC) ε G^{k-1}. Then at (IF,JF), I_{k-1}^k is just given by replacement: $(I_{k-1}^k v^{k-1})_{IF,JF} = v_{IC,JC}^{k-1}$. Suppose that at (IF+1,JF), L^k is given by the pointwise template

$$\begin{bmatrix} -NW & -N & -NE \\ -W & C & -E \\ -SW & -S & -SE \end{bmatrix} . \qquad (4.3)$$

Then

$$(I_{k-1}^k v^{k-1})_{IF+1,JF} =$$

$$\frac{(NW+W+SW)v_{IC,JC}^{k-1} + (NE+E+SE)v_{IC+1,JC}^{k-1}}{(C-N-S)} .$$

(We have just summed (4.3) vertically to average out its y-dependence.) A similar formula is used for points like (IF,JF+1). In each case, one performs $v^k \leftarrow v^k + I^k_{k-1} v^{k-1}$. Enough information is now present to use the difference equation at points like (IF+1,JF+1) to solve for $v^k_{IF+1,JF+1}$ in terms of its eight neighbors. Further details are contained in [5]. Since $10^{-4} \ll 1$, I^k_{k-1} is very nearly bilinear interpolation except near the boundary, where the above formulation gives a good extrapolation for points of $G^k \backslash G^{k-1}$ which do not lie between two points of G^{k-1} or in the center of four G^{k-1} points.

We take M = 4, IM = 2, IU = 1, IΓ =1, and we use $I^{k-1}_k = (I^k_{k-1})^*$ and $L^{k-1} = (I^k_{k-1})^* L^k I^k_{k-1}$. Then the asymptotic convergence factor per multigrid cycle for the multigrid algorithm described in Sec. 2 is .92. By asymptotic convergence factor we mean the ratio of the discrete L_2 norms of the residual on G^4 before and after a multigrid cycle $(G^4 \rightarrow G^3 \rightarrow G^2 \rightarrow G^1 \rightarrow G^2 \rightarrow G^3 \rightarrow G^4.)$

Now let us consider the ray multigrid method for (4.2). The set S is $\{(0,0),(\pi,\pi)\}$. Take $U^h_1 \equiv 1$ and $U^h_{-1} = (-1)^{(x+y)/h}$. Define $h = h_M$, $I^M_{M-1,i} = U^h_i I^M_{M-1}$, and $L^{M-1}_i = (I^M_{M-1,i})^* L^M I^M_{M-1,i}$, $i = -1,1$. Derive interpolation operators $I^{M-1}_{M-2,i}$ as described above from the L^{M-1}_i and define $I^{M-1}_{M-2,i,j} = U^h_j I^{M-1}_{M-2,i}$, $h = h_{M-1}$, $j = -1,1$. Continue recursively. (Thus, there are four $L^{M-2}_{i,j}$'s and eight $L^{M-3}_{i,j,k}$'s.) The asymptotic convergence factor per multigrid cycle for this algorithm is .21. Note that there is 3/2 as much storage and work per cycle for this algorithm versus the regular multigrid algorithm (since $1 + 1/4 + 1/16 + \ldots = 4/3$ and $1 + 2(1/4) + 4(1/16) + \ldots = 2$).

Another algorithm uses the corrections as soon as they are available instead of saving them up. This is done by using W-cycles and the diagram in Fig. 2 for M = 3, should make the algorithm clear. The asymptotic convergence factor for (4.2) per multigrid cycle (now a W-cycle) is .17.
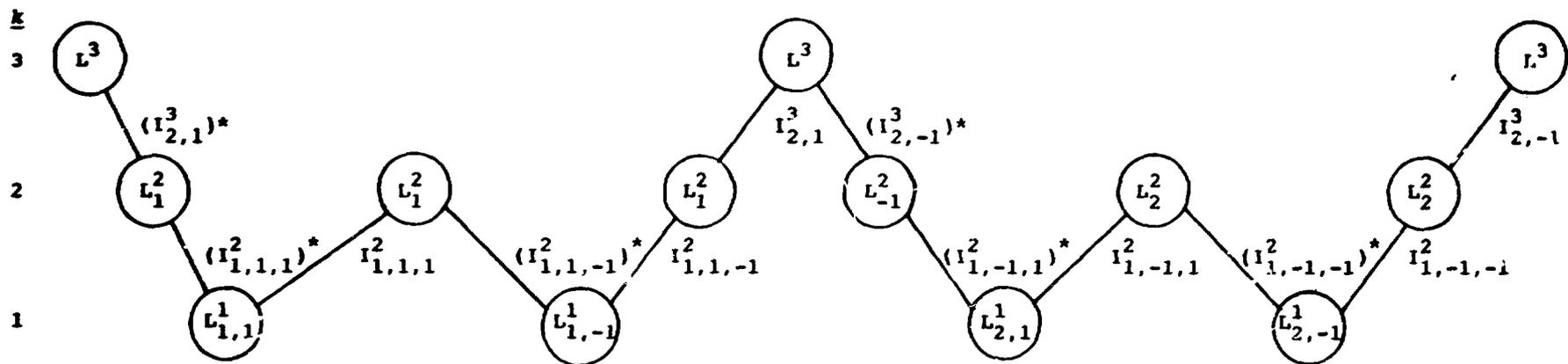
Fig. 2. W-cycle for ray multigrid with three grids.

5. THE RAY MULTIGRID METHOD FOR Δ_h^{sk} WITH ONLY ONE ARM.

As pointed out in [4], a matrix is never explicitly formed in the pressure iteration in SALE. Implementation of either method in the last section requires a matrix to be explicitly formed. What we want to investigate is whether these methods can be modified so as to be applicable without explicitly forming a matrix. As a first step, then, we consider whether L_s^k, $k < M$, $s \varepsilon S$, can be formed explicitly instead of from the variational (the $I^T LI$) approach.

First we recall the grid structure that was used in [4]. Instead of forming G^{k-1} by taking every other cell center of G^k, we let G^{k-1} be as in Fig. 3. We take I_{k-1}^k to be bilinear interpolation and $I_k^{k-1} = (I_{k-1}^k)^*$; for points near the boundary of G^k, we use fictitious cells and reflection (to approximate $\frac{\partial U}{\partial v} = 0$) to determine I_{k-1}^k; this gives rise to an extrapolation formula near the boundary.
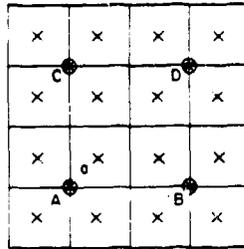

Fig. 3. Two cell-centered grids, in which coarse grids unknowns are not a subset of fine grid unknowns.

Consider first (4.2a) with the "finite element" boundary condition. Following [8], we seek an alternative definition of L_s^{M-1}, $s \varepsilon S$ so that

$$e^{is \cdot x/h} \, I_{M-1}^M L_s^{M-1}(e^{i2\theta \cdot x/2h}) \simeq L^M(e^{i(\theta+s) \cdot x/h}) \quad (5.1)$$

for small θ. For "finite element" boundary conditions we claim L^{M-1} can just be taken to be $L^{M-1} = -\Delta_h^{sk} \cdot + 10^{-4} \cdot$, $h = h_{M-1}$. Similarly we take $L^k = L_s^k = -\Delta_h^{sk} \cdot + 10 \cdot$, $h = h_k$. We take $I_{k-1,i}^k = U_i^h I_{k-1}^k$, $h = h_k$, and $I_{k,i}^{k-1} = (I_{k-1,k}^k)^*$, $i = -1,1$, and we use W-cycles. Analagous to Fig. 2, we now have Fig. 4. Also, following [8] we use the
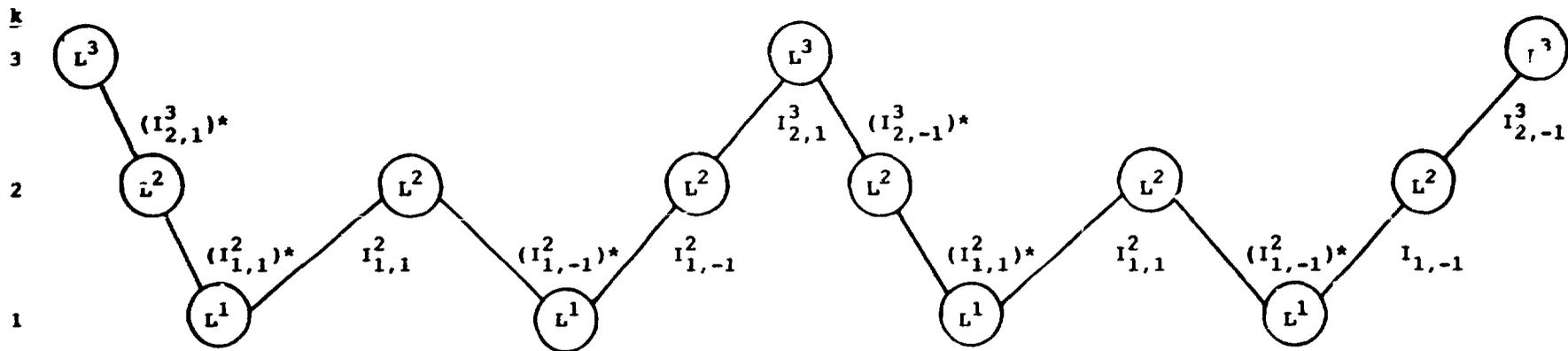
Fig. 4. W-cycle for ray multigrid with operator approximation and "finite element" boundary conditions.

smoothing operator S^{k-1} to smooth the solution on G^{k-1} before interpolating it to G^k, where

$$S^{k-1} = \frac{1}{8} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{5.2}$$

it is not necessary to do such smoothing in the variational approach, but we show by example below that it is necessary in the approaches of this section. We take $M = 3$, $IU = 1$, $ID = 1$, and $IM = 1$, so that there are two sweeps on G^3 for each W-cycle. The asymptotic convergence factor per W-cycle is 0.20. What should it be? The smoothing factor is

$$\mu = \max \left\{ \frac{-\frac{1}{2}e^{-i\theta_1}e^{i\theta_2} + \frac{1}{2}e^{i\theta_1}e^{i\theta_2}}{2 - \frac{1}{2}e^{-i\theta_1}e^{-i\theta_2} - \frac{1}{2}e^{i\theta_1}e^{-i\theta_2}} , \right.$$

$$\left. \frac{\pi}{2} \leq |\theta| \leq \pi \quad , \quad \frac{\pi}{2} \leq |\theta - (\pi,\pi)| \leq \pi \right\}$$

which is $\frac{1}{\sqrt{5}}$, assumed at $(\frac{\pi}{2},\pi)$. Since there are effectively two sweeps for each θ, per W-cycle, the convergence factor per W-cycle should be $(\frac{1}{\sqrt{5}})^2 = 0.2$. This crude analysis just happens to work in this case.

What happens when this method is applied with "finite difference" boundary conditions? The asymptotic convergence factor per W-cycle is at least 10^4. What is the cause of this divergence? With the "finite difference" boundary conditions, (5.1) is no longer true. Is there an approximation to $L_{-1}^{M-1} = (U_{-1}^h I_{M-1}^M)^* L^{M-1} U_{-1}^h I_{M-1}^M$, $h = h_M$, that can be made in the finite difference case that will cure divergence? An answer is provided by examining numerically the operators L_{-1}^{M-1}, etc. in the variational approach. L_{-1}^{M-1} looks as if it were derived from imposing Dirichlet boundary conditions; the same is

true for any L if U_{-1}^h, $h = h_j$, has been used in constructing constructing it or if one of its predecessors was constructed using U_{-1}^h, $h = h_k$, $k > j$. This suggests that Dirichlet boundary conditions should be imposed in the appropriate places in Fig. 4. Dirichlet "finite difference" boundary conditions can be derived by using fictitious cells, writing down a difference approximation to $U = 0$ (e.g., $\frac{1}{2}(U_{1,j}+U_{2,j}) = 0$, $2 \le j \le 25$) and then eliminating the fictitious cells in terms of the interior cells. Two typical cases are: at $(2,2)$, $_{-\Delta h}^{sk} U_{i,j} = \frac{1}{2h^2}(U_{i,j+1}+U_{i+1,j}-U_{i+1,j+1}+3U_{i,j})$ and at $(2,j)$, $2 < j < 25$, $-\Delta h^{sk} U_{i,j} = \frac{1}{2h^2}(U_{i,j-1}+U_{i,j+1}-U_{i+1,j+1}-U_{i+1,j-1}+4U_{i,j})$. We use the notation L_1^k and L_{-1}^k to denote $-\Delta_{h_k}^{sk} + 10^{-4}$ with zero Neumann and zero Dirichlet "finite difference" boundary conditions, respectively, and we have Fig. 5.

One remaining problem is that the computation of I^* is quite expensive; for each coarse grid cell center, I^* involves the weighted sum of sixteen residuals. In [4] what was used instead was

$$R_{k,\ell}^C = \frac{1}{4}(r_{i,j} + r_{i+1,j} + r_{i,j+1} + r_{i+1,j+1}) \quad , \quad (5.3)$$

where the $(k,\ell)^{\underline{th}}$ cell center on the coarse grid is in the center of the (i,j), $(i+1,j)$, $(i,j+1)$, $(i+1,j+1)$ fine cell centers. We use (5.3) here for $J_{3,1}^2$ and $J_{2,1}^1$, the replacements for $(I_{2,1}^3)^*$ and $(I_{1,1}^2)^*$. $J_{2,-1}^1$ and $J_{3,-1}^2$, the replacements for $I_{1,-1}^2$ and $I_{2,1}^3$ are given by

$$R_{k,\ell}^C = \frac{1}{4}(-r_{i,j} + r_{i+1,j} + r_{i,j+1} - r_{i+1,j+1}) \quad .$$

The asymptotic convergence factor per W-cycle with these changes instituted is 0.18.

Can we dispense with the smoothing operator s^{k-1} in this method? The result of doing so is an asymptotic convergence factor greater than 10^6. Since (5.2) thus appears to be so important, can we just use it coupled with the usual multigrid method (with no "crazy"

interpolations and operators)? The result of doing so is an asymptotic convergence factor (per V-cycle) of 0.93.

6. THE RAY MULTIGRID METHOD APPLIED TO THE PRESSURE ITERATION IN SALE.

A good question is why one should be interested in trying to accelerate the pressure iteration in SALE. If it has the bad feature of annihilating the (π,π) frequency, why not abandon it for something better? (An attempt was made to do exactly that in [4], but the resulting method was not better.) From a finite element point of view the method in SALE results from using piecewise bilinear elements and midpoint quadrature. If the method were implemented as a finite element method and four Gauss point quadrature were used, then the (π,π) frequency would not be annihilated; however, the calculation would be four times as expensive. In fact, one advocate of finite elements in fluid calculations confessed to me that he did not believe that his code could compete with the Lagrangian codes unless it used midpoint quadrature. On a given grid, the (π,π) frequency is badly approximated anyway, and while four Gauss point quadrature will give asymptotically more accurate answers, the goal of many fluid calculations is a qualitative representation of the flow; for such calculations, using midpoint quadrature may give accurate enough answers at less than one fourth the cost ("less than" since with midpoint quadrature the arithmetic simplifies considerably, a fact that SALE exploits to save storage by never explicitly forming a matrix). The "hourglass" instability can be controlled with smoothing of the (π,π) frequency - referred to as an "alternate node coupler" in SALE. One must exercise care not to introduce too much smoothing, and aside from philosophical qualms concerned with ad hoc smoothing, there is no reason to fault this procedure.

The paper [4] began with the application of multigrid to SOLA, an incompressible Eulerian code. For

the problem under consideration, we used the residual
weighting (5.3), which gave rise to a convergence factor
per work unit of 0.64 as opposed to the supposed possible
one of 0.595. Brandt devised another weighting scheme
which used (5.3) in the interior and another weighting
near the boundary. "Brandt's" weighting led to a
convergence factor of 0.595 per work unit and was thus
retained in the work for SOLA-ICE and SALE. However, in
computational studies for this paper, we discovered that
"Brandt's" weighting gave much worse results than (5.3)
for problems which discontinuous right-hand sides. On
the average, (5.3) appears to be better and is the
weighting which we now recommend. We accept the blame
for insufficient testing.

One problem for which (5.3) is much better than
"Brandt's" weighting is the Rayleigh-Taylor problem
worked with SALE in [4]. As commented in Sec. 1, we
changed the differencing in the pressure iteration in
SALE in [4]. Here we report on multigrid applied to SALE
using the original differencing in SALE and the residual
weighting (5.3). We take M = 3, G^3 = 12 cells × 12 cells
grid, IM = 2, IU = 2, and ID = 2; this is in contrast to
[4] where the accomodative mode of multigrid was used.
The other parameters are the same except for the sound
speed squared which is 2,000 in this paper and 20,000 in
[4]. On the coarsest grid we use the iteration of the
code coupled with the "constant addition" iteration; the
same is true for the single grid calculation; see [4] for
details. As in [4] we facilitate comparison between
single grid and multigrid by using the following time
steps:

$$\Delta t = \begin{cases} .5, & t < 15 \\ .25, & 15 \le t < 19 \\ .125, & 19 \le t < 23 \\ .0625, & 23 \le t < 25 \end{cases}.$$

No rezoning is used, and the grid distorts until a "bowtie" forms after t = 25. and the computation can no longer proceed; Fig. 6 shows a mesh near the end of the calculation.
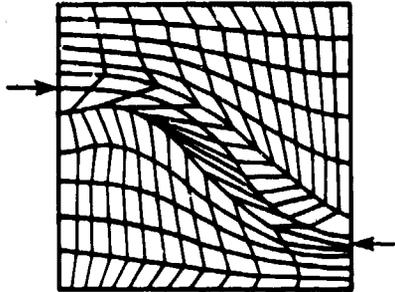

Fig. 6. Mesh used in Rayleigh-Taylor calculation at t=24.0.

(We remark that no "alternative node coupler" is used in this calculation. We tried using it, and it made little difference, as least for this problem. Also, we used an overrelaxation factor $\omega = 1.5$ for the single grid calculation as in [4].) Table 1 summarizes the comparison.

This is not exactly a smashing success for the multigrid method. Can the ray multigrid method be applied? One stumbling block is boundary conditions. The boundary conditions in SALE are applied by specifying u and v, the horizontal and vertical velocities. For the above problem u is specified on the top and bottom boundaries; this is equivalent to specifying $\frac{\partial p}{\partial v}$ on all boundaries. As remarked before, with $\frac{\partial p}{\partial v}$ specified on the boundary is $\Delta_h^{sk}p$ with the "finite difference" approximation to the boundary conditions. It can be checked that specifying v = 0 on the left and right boundaries and u = 0 on the bottom and top boundaries is equivalent to specifying p = 0 on all boundaries and that the approximation to the Laplacian with p = 0 specified on the boundary is $\Delta_h^{sk}p$ with the "finite difference" approximation to p = 0 on the boundary. Thus the algorithm displayed in Fig. 5 is implemented by specifying u[v] on the left and right boundaries and v[w] on the bottom and top boundaries for L_i^k if i = 1[-1].

Table 1. Comparison of single grid and regular multigrid in SALE.

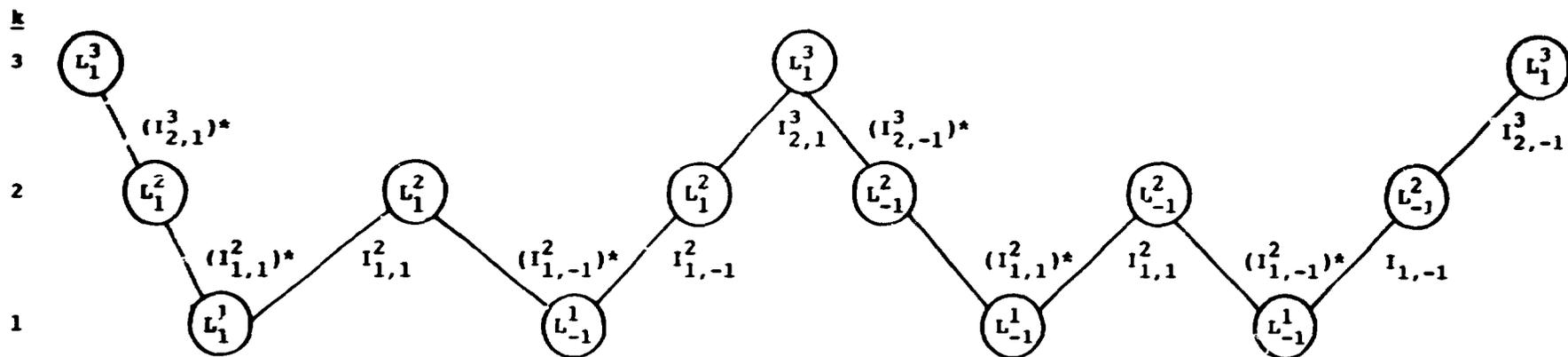| Time	Total time spent iterating (MG/SG)	Total calculational time (MG/SG)	Fraction of calculation spent iterating (SG)	Fraction of calculation spent iterating (MG)	Convergence factor per work unit on last relaxation sweep, this time step (SG)	Convergence factor per work unit* on last cycle, this time step (MG)
t = 3.0	.96	.76	.63	.63	.82	.77
t = 15.0	.76	.77	.69	.66	.82	.67
t = 19.0	.69	.73	.76	.74	.85	.80
t = 23.0	.76	.79	.77	.76	.69	.80
t = 26.0	.83	.64	.79	.79	.84	.86
t = 25.0	.88	.89	.68	.69	.82	.89

* Only relaxation work is counted.

Fig. 5. W-cycle for ray multigrid with operator approximation and "finite difference" boundary conditions.

The constant addition algorithm is used on grid 1 only for L_1^1. We take IM = 2, IU = 2, ID = 2. The results, in Table 2, are promising for such a small problem. The convergence factor for the single grid calculation can be expected to increase with the number of unknowns, whereas the multigrid convergence factor should remain bounded. We avoided rezoning in this calculation to exhibit that multigrid can still function well on highly distorted grids.

The experience here is the same as in all other successful applications of multigrid to hard problems with which we are familiar, and that is that the way to success is torturous and full of pitfalls for the unwary. One can only hope that as experience is gained and knowledge accumulated, the way will become easier.

Table 2. Comparison of single grid and ray multgrid in SALE.

Time	Total time spent iterating (MG/SG)	Total calculational time (MG/SG)	Fraction of calculation spent iterating (MG)	Convergence factor per W-cycle and work unit* on last cycle, this time step (MG)
t = 3.0	.64	.60	.55	.05, .64
t = 15.0	.61	.67	.58	.06, .67
t = 19.0	.60	.66	.62	.17, .77
t = 23.0	.66	.71	.71	.19, .79
t = 24.0	.68	.72	.73	.10, .72
t = 25.0	.68	.72	.74	.14, .75

* Only relaxation work is counted.

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Theoretical Division, MS-B284
Los Alamos National Laboratory
Los Alamos, NM 87545